

CONGRUENCES ARISING FROM APÉRY-TYPE SERIES FOR ZETA VALUES

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ABSTRACT. Recently, R. Tauraso established finite p -analogues of Apéry's famous series for $\zeta(2)$ and $\zeta(3)$. In this paper, we present several congruences for finite central binomial sums arising from the truncation of Apéry-type series for $\zeta(4)$ and $\zeta(5)$. We also prove a p -analogue of Zeilberger's series for $\zeta(2)$ confirming a conjecture of Z. W. Sun.

1. INTRODUCTION

The Riemann zeta function for $\operatorname{Re} s > 1$ is defined by the series $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$. It is well-known, due to Euler, that the value of the Riemann zeta function at an even positive integer can be expressed in terms of π . Namely,

$$\zeta(2m) = (-1)^{m-1} (2\pi)^{2m} \frac{B_{2m}}{2(2m)!}, \quad m \in \mathbb{N},$$

where $B_m \in \mathbb{Q}$ are Bernoulli numbers defined by the series expansion

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!},$$

which yields $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_{2m+1} = 0$, $m \in \mathbb{N}$.

The alternating multiple harmonic sum is defined as

$$H_n(a_1, a_2, \dots, a_r) = \sum_{1 \leq k_1 < k_2 < \dots < k_r \leq n} \prod_{i=1}^r \frac{\operatorname{sgn}(a_i)^{k_i}}{k_i^{|a_i|}},$$

where $n \geq 0$, $r \geq 1$ and $(a_1, a_2, \dots, a_r) \in (\mathbb{Z}^*)^r$ (here and in the sequel an empty sum is treated as zero).

Recently, R. Tauraso [13] showed that Apéry's famous series for $\zeta(3)$ and $\zeta(2)$,

$$(1) \quad \zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \quad \text{and} \quad \zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}},$$

that were used in his irrationality proofs [9] of these numbers admit very nice p -analogues:

$$(2) \quad \frac{5}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \equiv \frac{H_{p-1}(1)}{p^2} \quad \text{and} \quad 3 \sum_{k=1}^{p-1} \frac{1}{k^2 \binom{2k}{k}} \equiv \frac{H_{p-1}(1)}{p} \pmod{p^3},$$

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where $p > 5$ is a prime. The similar Apéry-type series for $\zeta(4)$ is also well-known (see [3, p. 89]):

$$(3) \quad \zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}.$$

A generalization of formulas (1), (3) to odd zeta values $\zeta(2n+3)$, $n \in \mathbb{N}$, with the help of the generating function identity

$$(4) \quad \sum_{n=0}^{\infty} \zeta(2n+3) a^{2n} = \sum_{k=1}^{\infty} \frac{1}{k(k^2 - a^2)} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \frac{5k^2 - a^2}{k^2 - a^2} \prod_{m=1}^{k-1} \left(1 - \frac{a^2}{m^2}\right),$$

where $a \in \mathbb{C}$, $|a| < 1$, was given by Koecher [6] (and independently in an expanded form by Leshchiner [7]). Expanding the right-hand side of (4) in powers of a^2 and comparing coefficients of a^{2n} on both sides of (4) gives the Apéry-like series for $\zeta(2n+3)$. In particular, comparing constant terms recovers formula (1) for $\zeta(3)$ and comparing coefficients of a^2 gives the following formula for $\zeta(5)$:

$$(5) \quad \zeta(5) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} H_{k-1}(2)}{k^3 \binom{2k}{k}}.$$

First results related to generating function identities for even zeta values belong to Leshchiner [7] who proved (in an expanded form) that for $|a| < 1$,

$$(6) \quad \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{2n+1}}\right) \zeta(2n+2) a^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 - a^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \frac{3k^2 + a^2}{k^2 - a^2} \prod_{m=1}^{k-1} \left(1 - \frac{a^2}{m^2}\right).$$

Comparing constant terms on both sides of (6) implies formula (1) for $\zeta(2)$ and comparing coefficients of a^2 yields

$$(7) \quad \zeta(4) = \frac{16}{7} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} - \frac{12}{7} \sum_{k=1}^{\infty} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}}.$$

From (3), (7) we get easily the following reduction formula:

$$\sum_{k=1}^{\infty} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} = \frac{5}{51} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{5}{108} \zeta(4).$$

In 2006, D. Bailey, J. Borwein and D. Bradley [2] proved another identity

$$(8) \quad \sum_{n=0}^{\infty} \zeta(2n+2) a^{2n} = \sum_{k=1}^{\infty} \frac{1}{k^2 - a^2} = 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k} (k^2 - a^2)} \prod_{m=1}^{k-1} \left(\frac{m^2 - 4a^2}{m^2 - a^2}\right).$$

It generates similar Apéry-like series for the numbers $\zeta(2n+2)$, which are not covered by Leshchiner's result (6). In particular, for $\zeta(4)$ it gives

$$(9) \quad \zeta(4) = 3 \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} - 9 \sum_{k=1}^{\infty} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}}.$$

In this paper, we prove p -analogues of Apéry-type series for $\zeta(4)$ and $\zeta(5)$ arising from the truncation of the series (3), (5), (7) and (9).

Theorem 1. *Let $p > 3$ be a prime. Then we have*

$$4 \sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - 3 \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \equiv \frac{3}{p^3} H_{p-1}(1) - \frac{6}{5} p B_{p-5} \pmod{p^2}.$$

Theorem 2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} \equiv \frac{H_{p-1}(1)}{p^3} \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv \frac{-2H_{p-1}(1)}{p^2} \pmod{p}.$$

In [12, Conj. 1.1] Z. W. Sun conjectured that for each prime $p > 5$,

$$(10) \quad \sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} \equiv \frac{H_{p-1}(1)}{p^3} - \frac{7}{45} p B_{p-5} \pmod{p^2}$$

and for $p > 7$,

$$(11) \quad \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv \frac{-2H_{p-1}(1)}{p^2} - \frac{13}{27} H_{p-1}(3) \pmod{p^4}.$$

Theorem 2 confirms (10), (11) modulo a prime. Moreover, from Theorem 1 it follows that (10) is equivalent to the following

Conjecture 1. *Let $p > 3$ be a prime. Then*

$$(12) \quad \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \equiv \frac{H_{p-1}(1)}{3p^3} + \frac{26}{135} p B_{p-5} \pmod{p^2}.$$

In this paper we prove (12) modulo a prime (see Lemma 1 below).

Corollary 1. *Let $p > 5$ be a prime. Then we have*

$$\sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - 3 \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \equiv 0 \pmod{p}.$$

Theorem 3. *Let $p > 3$ be a prime. Then we have*

$$2 \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1} H_{k-1}(2)}{k^3 \binom{2k}{k}} \equiv -\frac{6}{5} B_{p-5} \pmod{p}.$$

Theorem 4. *Let $p > 5$ be a prime. Then*

$$p \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} \equiv \frac{5(1 - L_p^2)}{8p^4} - \frac{5}{4p^3} (\mathcal{L}_1(\varphi^2) + \mathcal{L}_1(\varphi^{-2})) + \frac{3H_{p-1}(1)}{p^3} - \frac{3p}{5} B_{p-5} \pmod{p^2},$$

and

$$\sum_{k=1}^{p-1} \frac{(-1)^k \binom{2k}{k}}{k^4} \equiv \frac{5(1 - L_p^2)}{4p^4} - \frac{5}{2p^3} (\mathcal{L}_1(\varphi^2) + \mathcal{L}_1(\varphi^{-2})) + \frac{6H_{p-1}(1)}{p^3} \pmod{p},$$

where L_n is the n th Lucas number defined by the recurrence $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $n > 1$, $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\mathcal{L}_1(x) = \sum_{k=1}^{p-1} x^k/k$ is the finite 1-logarithm.

The first congruence of Theorem 4 gives a finite p -analogue of the following alternating central binomial sum that was evaluated explicitly in [1]:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} = -2\zeta(5) + \frac{5}{2} \text{Li}_5(\varphi^{-2}) + 5 \text{Li}_4(\varphi^{-2}) \log \varphi + 4\zeta(3) \log^2 \varphi - \frac{8}{3} \zeta(2) \log^3 \varphi + \frac{4}{3} \log^5 \varphi,$$

where $\text{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$ is the classical polylogarithm.

In [8], S. Mattarei and R. Tauraso extended results of [13] and described a general approach on obtaining congruences for the finite sums

$$(13) \quad \sum_{k=1}^{p-1} \frac{t^k}{k^d \binom{2k}{k}} \pmod{p^2}, \quad d = 0, 1, 2, 3^1,$$

and

$$(14) \quad p \sum_{k=1}^{p-1} \frac{H_{k-1}(2) t^k}{k^d \binom{2k}{k}} \pmod{p}, \quad d = 0, 1, 2.$$

Note that some special cases of sum (14) were considered earlier by Z. W. Sun in [11]. The crucial idea of work [8] relies on a connection of values (13), (14) with sums of the form

$$\sum_{k=1}^{p-1} \frac{u_k(2-t)}{k^d} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{v_k(2-t)}{k^d}$$

that can be written in terms of the finite polylogarithms (see [8, §8])

$$\mathcal{L}_d(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^d}, \quad d \in \mathbb{N}.$$

Here $\{u_n(x)\}_{n \geq 0}$ and $\{v_n(x)\}_{n \geq 0}$ are Lucas sequences defined by the recurrence relations

$$\begin{aligned} u_0(x) &= 0, \quad u_1(x) = 1, \quad \text{and} \quad u_n(x) = xu_{n-1}(x) - u_{n-2}(x) \quad \text{for } n > 1, \\ v_0(x) &= 2, \quad v_1(x) = x, \quad \text{and} \quad v_n(x) = xv_{n-1}(x) - v_{n-2}(x) \quad \text{for } n > 1. \end{aligned}$$

¹when $d = 3$ the congruence was established only modulo p (see [8, Cor. 6.4])

In principle, their method can be generalized to get congruences for sums (13) with $d \geq 4$. So, for example, R. Tauraso communicated us the following identity:

$$\begin{aligned} \binom{2n}{n} \sum_{k=1}^n \frac{t^k}{k^4 \binom{2k}{k}} &= \sum_{k=1}^n \binom{2n}{n-k} \frac{v_k(t-2)}{k^4} + \binom{2n}{n} \sum_{k=1}^n \frac{1}{k^4} \\ &\quad + 2 \sum_{1 \leq j < k \leq n} \binom{2n}{n-k} \left(\frac{1}{k} + \frac{1}{j} \right) \frac{(-1)^{k-j} v_j(t-2)}{jk^2} \\ &\quad + 4 \sum_{1 \leq i < j < k \leq n} \binom{2n}{n-k} \frac{(-1)^{k-i} v_i(t-2)}{ijk^2} \end{aligned}$$

that can be obtained by integration from [8, Th. 5.3]. In the easiest case, when $t = 4$, then $v_n(t-2) = 2$ for $n \geq 0$ and it follows that (see [8, Section 8])

$$p \sum_{k=1}^{p-1} \frac{4^k}{k^4 \binom{2k}{k}} \equiv -\frac{4}{3} (2q_p(2)^3 + B_{p-3}) \pmod{p},$$

where $q_p(2) = \frac{2^{p-1}-1}{p}$ is the Fermat quotient. For other values of t , in particular, for $t = 1$ it is not so easy to derive similar congruences, since one needs evaluations for values of finite multiple polylogarithms modulo a power of a prime. To prove our Theorems 1–4, we employ another method based on application of appropriate WZ pairs that were found in [4] for demonstrating identities (4), (6), (8). The similar approach also allows us to establish the following p -analogue of Zeilberger's series for $\zeta(2)$,

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{21k-8}{k^3 \binom{2k}{k}^3}.$$

Theorem 5. *Let p be a prime greater than 5. Then we have*

$$\sum_{k=1}^{p-1} \frac{21k-8}{k^3 \binom{2k}{k}^3} + \frac{p-1}{p^3} \equiv \frac{H_{p-1}(1)}{p^2} (15p-6) + \frac{12}{5} p^2 B_{p-5} \pmod{p^3}.$$

Note that Theorem 5 confirms Z. W. Sun's conjecture [12, (1.19)].

The paper is organized as follows. In Sections 2 and 3, we recall some important divisibility properties of multiple harmonic sums and prove some helpful lemmas. In Section 4, we give proofs of Theorems 1–4. In final Section 5, we demonstrate some interesting combinatorial identities and interrelated congruences that are essential for the proof of Theorem 5.

2. MULTIPLE HARMONIC SUMS

We start by recalling some important properties of multiple harmonic sums. Let a, b, c be positive integers. We will need the following two formulas for the product:

$$H_n(a) \cdot H_n(b) = H_n(a, b) + H_n(b, a) + H_n(a+b),$$

$$H_n(a, b) \cdot H_n(c) = H_n(a, b, c) + H_n(a, c, b) + H_n(c, a, b) + H_n(a+c, b) + H_n(a, b+c).$$

The divisibility properties of multiple harmonic sums were studied in [5, 10], [13]–[16] and many of them are related to the Bernoulli numbers:

(a) ([10, Th. 5.1, Cor. 5.1]) for $a > 0$ and for any prime $p \geq a + 3$,

$$H_{p-1}(a) \equiv \begin{cases} -\frac{a(a+1)}{2(a+2)} p^2 B_{p-a-2} \pmod{p^3} & \text{if } a \text{ is odd,} \\ \frac{a}{a+1} p B_{p-a-1} \pmod{p^2} & \text{if } a \text{ is even;} \end{cases}$$

(b) ([10, Rem. 5.1], [13, Th. 2.1]) for any prime $p > 5$,

$$\begin{aligned} H_{p-1}(1) &\equiv p^2 \left(2 \frac{B_{p-3}}{p-3} - \frac{B_{2p-4}}{2p-4} \right) \pmod{p^4}, \\ H_{p-1}(2) &\equiv -\frac{2H_{p-1}(1)}{p} + \frac{2}{5} p^3 B_{p-5} \pmod{p^4}, \\ H_{p-1}(1) &\equiv p^2 \left(\frac{B_{3p-5}}{3p-5} - 3 \frac{B_{2p-4}}{2p-4} + 3 \frac{B_{p-3}}{p-3} \right) + p^4 \frac{B_{p-5}}{p-5} \pmod{p^5}. \end{aligned}$$

(c) ([16], [15, Th. 1.6]) for $a, r > 0$ and for any prime $p > ar + 2$,

$$H_{p-1}(\{a\}^r) \equiv \begin{cases} (-1)^r \frac{a(ar+1)p^2}{2(ar+2)} B_{p-ar-2} \pmod{p^3} & \text{if } ar \text{ is odd,} \\ (-1)^{r-1} \frac{ap}{ar+1} B_{p-ar-1} \pmod{p^2} & \text{if } ar \text{ is even;} \end{cases}$$

(d) ([15, Th. 3.1, 3.2]) for $a_1, a_2 > 0$ and for any prime $p \geq a_1 + a_2$,

$$H_{p-1}(a_1, a_2) \equiv \frac{(-1)^{a_2}}{a_1 + a_2} \binom{a_1 + a_2}{a_1} B_{p-a_1-a_2} \pmod{p},$$

moreover, if $a_1 + a_2$ is even, then for any prime $p > a_1 + a_2 + 1$,

$$\begin{aligned} H_{p-1}(a_1, a_2) &\equiv p \left[(-1)^{a_1} a_2 \binom{a_1 + a_2 + 1}{a_1} - (-1)^{a_1} a_1 \binom{a_1 + a_2 + 1}{a_2} - a_1 - a_2 \right] \\ &\quad \times \frac{B_{p-a_1-a_2-1}}{2(a_1 + a_2 + 1)} \pmod{p^2}; \end{aligned}$$

(e) ([15, Th. 3.5]) if $a_1, a_2, a_3 > 0$ and $w := a_1 + a_2 + a_3$ is odd, then for any prime $p > a_1 + a_2 + a_3$,

$$H_{p-1}(a_1, a_2, a_3) \equiv \left[(-1)^{a_1} \binom{w}{a_1} - (-1)^{a_3} \binom{w}{a_3} \right] \frac{B_{p-w}}{2w} \pmod{p};$$

(f) ([15, Prop. 3.8], [5, Th. 5.2, 7.2]) for any prime $p > 5$,

$$H_{p-1}(1, 1, 2) \equiv \frac{11}{10} p B_{p-5}, \quad H_{p-1}(1, 2, 1) \equiv -\frac{9}{10} p B_{p-5} \pmod{p^2},$$

$$H_{p-1}(2, 1, 1) \equiv \frac{3}{5} p B_{p-5} \pmod{p^2}, \quad H_{p-1}(1, 1, 1, 2) \equiv H_{p-1}(1, 4) \equiv B_{p-5} \pmod{p};$$

(g) ([14, Cor. 2.3, Prop. 7.3, 6.1]) for any prime $p > 5$,

$$\begin{aligned} H_{p-1}(-4) &\equiv \frac{3}{4}pB_{p-5}, & H_{p-1}(-3) &\equiv -2H_{p-1}(1, -2) \equiv \frac{3}{2}\frac{H_{p-1}(1)}{p^2} \pmod{p^2}, \\ H_{p-1}(2, -2) &\equiv -2H_{p-1}(1, -3), & 2H_{p-1}(1, 1, -2) &\equiv H_{p-1}(1, -3) \pmod{p}. \end{aligned}$$

Note that by (a), (b), we can replace the harmonic number $H_{p-1}(1)$ appearing in the congruences of Theorems 1, 2, 4, 5 by an appropriate expression in terms of Bernoulli numbers.

3. PRELIMINARIES

Lemma 1. *Let p be a prime greater than 5. Then*

$$(15) \quad p \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \equiv \frac{1}{3} \frac{H_{p-1}(1)}{p^2} \pmod{p^2}.$$

Proof. Note that this congruence modulo p easily follows from Lemma 6.1 of [8]. Following the same scheme of the proof and applying (2), it is possible to derive the more exact congruence modulo p^2 . We reproduce the proof here for completeness. In [13, Th. 3.1] it was shown that

$$(16) \quad \sum_{k=0}^n \frac{\binom{n}{k} \binom{n+k-1}{k}}{\binom{2k}{k}} (-t)^k = \frac{(-1)^n v_n(t-2)}{2},$$

where $\{v_n(x)\}_{n \geq 0}$ is the Lucas sequence defined in the Introduction. Taking $n = p$, where p is an odd prime, and noting that $v_n(x) = (-1)^n v_n(-x)$, we get

$$(17) \quad \sum_{k=0}^p \frac{\binom{p}{k} \binom{p+k-1}{k}}{\binom{2k}{k}} (-t)^k = \frac{v_p(2-t)}{2}.$$

For $1 \leq k \leq p-1$, we have

$$(18) \quad (-1)^{k-1} \binom{p}{k} \binom{p+k-1}{k} = \frac{p^2}{k^2} \prod_{m=1}^{k-1} \left(1 - \frac{p^2}{m^2}\right) \equiv \frac{p^2}{k^2} (1 - p^2 H_{k-1}(2)) \pmod{p^6}.$$

Now, since $\binom{2k}{k}$, for $p/2 < k < p$, is a multiple of p but not of p^2 , from (17), (18) we obtain

$$p^4 \sum_{k=1}^{p-1} \frac{t^k H_{k-1}(2)}{k^2 \binom{2k}{k}} \equiv p^2 \sum_{k=1}^{p-1} \frac{t^k}{k^2 \binom{2k}{k}} + \frac{v_p(2-t) + t^p - 2}{2} \pmod{p^5}.$$

Setting in the above congruence $t = 1$ and taking into account (2) we get for $p > 5$,

$$p \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \equiv \frac{H_{p-1}(1)}{3p^2} + \frac{v_p(1) - 1}{p^3} \pmod{p^2}.$$

Now noting that $v_p(1) = \alpha^p + \alpha^{-p}$, where α is a root of the polynomial $x^2 - x + 1$ and p is a prime greater than 3, i.e., $v_p(1) = 2 \cos(\pi p/3)$ and $p \equiv \pm 1 \pmod{6}$, we get $v_p(1) = 1$, and the lemma follows. \square

Lemma 2. *Let p be a prime greater than 5. Then*

$$p \sum_{k=1}^{p-1} \frac{(-1)^k H_{k-1}(2)}{k^3 \binom{2k}{k}} \equiv \frac{L_p^2 - 1}{2p^4} + \frac{1}{p^3} \left(\mathcal{L}_1(\varphi^2) + \mathcal{L}_1(\varphi^{-2}) \right) - \frac{12H_{p-1}(1)}{5p^3} \pmod{p^2},$$

where L_n is the n th Lucas number defined by the recurrence $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $n > 1$, $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\mathcal{L}_1(x) = \sum_{k=1}^{p-1} x^k/k$ is the finite 1-logarithm.

Proof. Rewrite identity (16) in the form

$$\sum_{k=1}^n \frac{\binom{n}{k} \binom{n+k-1}{k}}{\binom{2k}{k}} (-t)^k = \frac{v_n(2-t) - 2}{2}.$$

Dividing both sides by t and integrating with respect to t , we see from the relation (see [8, Lemma 5.1])

$$\int_0^t \frac{v_n(2-t) - 2}{t} dt = \frac{v_n(2-t) - 2}{n} + 2 \sum_{k=1}^{n-1} \frac{v_k(2-t) - 2}{k}$$

that

$$(19) \quad \sum_{k=1}^n \frac{\binom{n}{k} \binom{n+k-1}{k}}{\binom{2k}{k}} \frac{(-t)^k}{k} = \frac{v_n(2-t) - 2}{2n} + \sum_{k=1}^{n-1} \frac{v_k(2-t) - 2}{k}.$$

Now taking $n = p$ in (19), by (18), we obtain

$$\begin{aligned} p^4 \sum_{k=1}^{p-1} \frac{t^k H_{k-1}(2)}{k^3 \binom{2k}{k}} &\equiv p^2 \sum_{k=1}^{p-1} \frac{t^k}{k^3 \binom{2k}{k}} + \frac{v_p(2-t) - 2 + t^p}{2p} \\ &\quad + \sum_{k=1}^{p-1} \frac{v_k(2-t)}{k} - 2H_{p-1}(1) \pmod{p^5}. \end{aligned}$$

Setting $t = -1$ in the above congruence and employing (2) we deduce

$$p \sum_{k=1}^{p-1} \frac{(-1)^k H_{k-1}(2)}{k^3 \binom{2k}{k}} \equiv \frac{v_p(3) - 3}{2p^4} + \frac{1}{p^3} \sum_{k=1}^{p-1} \frac{v_k(3)}{k} - \frac{12H_{p-1}(1)}{5p^3} \pmod{p^2}.$$

Since $v_k(3) = \varphi^{2k} + \varphi^{-2k} = L_{2k} = L_k^2 + 2$, we conclude the proof. \square

Observing that $\mathcal{L}_1(\varphi^2) + \mathcal{L}_1(\varphi^{-2}) = \sum_{k=1}^{p-1} L_{2k}/k$ and

$$p^2 \sum_{k=1}^{p-1} \frac{(-1)^k H_{k-1}(2)}{k^3 \binom{2k}{k}} \equiv 0 \pmod{p}$$

we get the following interesting corollary.

Corollary 2. *Let p be an odd prime and L_n be the n th Lucas number. Then we have*

$$\sum_{k=1}^{p-1} \frac{L_{2k}}{k} \equiv \frac{1 - L_p^2}{2p} + \frac{12}{5} H_{p-1}(1) \pmod{p^3}.$$

(Note that $L_p \equiv 1 \pmod{p}$.)

The following lemma refines the corresponding result from [13, Th. 2.3].

Lemma 3. *Let $p > 3$ be a prime. Then*

$$(20) \quad H_{p-1}(1, 2) \equiv \frac{6}{5}p^2 B_{p-5} - H_{p-1}(2, 1) \equiv -3 \frac{H_{p-1}(1)}{p^2} + \frac{1}{2}p^2 B_{p-5} \pmod{p^3}.$$

Proof. The first congruence in (20) easily follows from the identity

$$(21) \quad H_k(1)H_k(2) = H_k(1, 2) + H_k(2, 1) + H_k(3).$$

Indeed, by (a) we have that for any prime $p > 3$, $H_{p-1}(1) \equiv 0 \pmod{p^2}$, $H_{p-1}(2) \equiv 0 \pmod{p}$ and for $p > 5$, $H_{p-1}(3) \equiv -\frac{6}{5}p^2 B_{p-5} \pmod{p^3}$, which implies

$$H_{p-1}(2, 1) + H_{p-1}(1, 2) \equiv \frac{6}{5}p^2 B_{p-5} \pmod{p^3}, \quad p > 5.$$

To prove the second congruence in (20), we consider the following identity (see [13, proof of Th. 2.3]):

$$\sum_{k=1}^n \frac{1}{k^2} = \sum_{1 \leq i \leq j \leq n} \frac{(-1)^{j-1}}{ij} \binom{n}{j}, \quad n \in \mathbb{N}.$$

Setting $n = p$ we get

$$\begin{aligned} H_{p-1}(2) &= p \sum_{1 \leq i \leq j \leq p-1} \frac{(-1)^{j-1}}{ij^2} \binom{p-1}{j-1} + \frac{H_{p-1}(1)}{p} \\ &\equiv p \sum_{1 \leq i \leq j \leq p-1} \frac{1 - pH_{j-1}(1) + p^2 H_{j-1}(1, 1)}{ij^2} + \frac{H_{p-1}(1)}{p} \\ &\equiv pH_{p-1}(3) + pH_{p-1}(1, 2) + \frac{H_{p-1}(1)}{p} - p^2 H_{p-1}(1, 3) - p^2 \sum_{1 \leq i < j \leq p-1} \frac{H_{j-1}(1)}{ij^2} \\ &\quad + p^3 H_{p-1}(1, 1, 3) + p^3 \sum_{1 \leq i < j \leq p-1} \frac{H_{j-1}(1, 1)}{ij^2} \pmod{p^4}. \end{aligned}$$

Since

$$\sum_{1 \leq i < j \leq p-1} \frac{H_{j-1}(1)}{ij^2} = \sum_{j=1}^{p-1} \frac{H_{j-1}^2(1)}{j^2} \quad \text{and} \quad \sum_{1 \leq i < j \leq p-1} \frac{H_{j-1}(1, 1)}{ij^2} = \sum_{j=1}^{p-1} \frac{H_{j-1}(1, 1)H_{j-1}(1)}{j^2},$$

then by the formulas

$$(22) \quad H_n^2(1) = 2H_n(1, 1) + H_n(2)$$

and

$$H_n(1, 1)H_n(1) = 3H_n(1, 1, 1) + H_n(2, 1) + H_n(1, 2),$$

we get

$$\begin{aligned}
 (23) \quad H_{p-1}(2) &\equiv pH_{p-1}(3) + pH_{p-1}(1, 2) + \frac{H_{p-1}(1)}{p} - p^2 H_{p-1}(1, 3) \\
 &\quad - 2p^2 H_{p-1}(1, 1, 2) - p^2 H_{p-1}(2, 2) + p^3 H_{p-1}(1, 1, 3) + 3p^3 H_{p-1}(1, 1, 1, 2) \\
 &\quad + p^3 H_{p-1}(2, 1, 2) + p^3 H_{p-1}(1, 2, 2) \pmod{p^4}.
 \end{aligned}$$

Now we can evaluate the right-hand side modulo p^4 using known congruences for multiple harmonic sums. By (c)–(e), for any prime $p > 5$ we have

$$(24) \quad H_{p-1}(2, 2) \equiv -\frac{2}{5}pB_{p-5}, \quad H_{p-1}(1, 3) \equiv -\frac{9}{10}pB_{p-5} \pmod{p^2}$$

and

$$H_{p-1}(1, 2, 2) \equiv -\frac{3}{2}B_{p-5}, \quad H_{p-1}(2, 1, 2) \equiv 0, \quad H_{p-1}(1, 1, 3) \equiv \frac{1}{2}B_{p-5} \pmod{p}.$$

Further by (b) and (f), substituting the above congruences in (23) and simplifying we conclude that for $p > 5$,

$$H_{p-1}(1, 2) \equiv -3\frac{H_{p-1}(1)}{p^2} + \frac{1}{2}p^2 B_{p-5} \pmod{p^3}.$$

The validity of (20) for $p = 5$ can be easily checked by hand. \square

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.

For any non-negative integers n, k , consider the pair of functions

$$\begin{aligned}
 (25) \quad F(n, k) &= \frac{(-1)^{n+k}(n-k-1)!k!^2}{(n+k+1)!} H_k(2), \quad n \geq k+1, \\
 G(n, k) &= \frac{2(-1)^{n+k}(n-k)!k!^2}{(n+k+1)!(n+1)} \left(H_k(2) - \frac{1}{(n+1)^2} \right), \quad n \geq k.
 \end{aligned}$$

By straightforward verification it is easy to check that (F, G) is a WZ pair, i.e.,

$$(26) \quad F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k)$$

for any $n, k \geq 0$, $n \geq k+1$. Now putting $h(n) := \sum_{k=0}^{n-1} F(n, k)$, $n \geq 1$, and summing (26) over $k = 0, 1, \dots, n-1$ we obtain

$$(27) \quad h(n+1) - h(n) = G(n, n) + F(n+1, n) - G(n, 0).$$

Again, summing (27) over $n = 1, 2, \dots, N$ we get

$$h(N+1) - h(1) = \sum_{n=1}^N (G(n, n) + F(n+1, n)) - \sum_{n=1}^N G(n, 0)$$

which is equivalent to the following summation formula:

$$(28) \quad \sum_{n=0}^N G(n, 0) = \sum_{n=0}^N (G(n, n) + F(n+1, n)) - \sum_{k=0}^N F(N+1, k).$$

Now substituting the WZ pair (F, G) defined by (25) in (28), simplifying and replacing N by $N - 1$ we get the identity

$$4 \sum_{k=1}^N \frac{1}{k^4 \binom{2k}{k}} = 3 \sum_{k=1}^N \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} - 2 \sum_{k=1}^N \frac{(-1)^k}{k^4} + \sum_{k=1}^N \frac{(-1)^{N+k} (N-k)! (k-1)!^2 H_{k-1}(2)}{(N+k)!}.$$

Setting $N = p - 1$ and observing that

$$\begin{aligned} & (-1)^k \frac{(p-1-k)!(k-1)!^2}{(p-1+k)!} = \frac{1}{pk} \prod_{m=1}^k \left(1 - \frac{p}{m}\right)^{-1} \prod_{m=1}^{k-1} \left(1 + \frac{p}{m}\right)^{-1} \\ & \equiv \frac{1}{pk} \left(1 + pH_k(1) + p^2(H_k^2(1) - H_k(1, 1))\right) \left(1 - pH_{k-1}(1) + p^2(H_{k-1}^2(1) - H_{k-1}(1, 1))\right) \\ & \equiv \frac{1}{pk} + \frac{1}{k^2} + \frac{pH_k(2)}{k} \pmod{p^2}, \end{aligned}$$

where in the last congruence we used (22), we obtain

$$\begin{aligned} & 4 \sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - 3 \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \\ & \equiv -2H_{p-1}(-4) + \frac{1}{p}H_{p-1}(2, 1) + H_{p-1}(2, 2) + p \sum_{k=1}^{p-1} \frac{H_{k-1}(2)H_k(2)}{k} \pmod{p^2}. \end{aligned}$$

Applying to the last sum the identity

$$(H_k(2))^2 = 2H_k(2, 2) + H_k(4)$$

we get

$$\begin{aligned} & 4 \sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - 3 \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k^2 \binom{2k}{k}} \\ (30) \quad & \equiv -2H_{p-1}(-4) + \frac{1}{p}H_{p-1}(2, 1) + H_{p-1}(2, 2) \\ & + pH_{p-1}(2, 3) + 2pH_{p-1}(2, 2, 1) + pH_{p-1}(4, 1) \pmod{p^2}. \end{aligned}$$

Now from (d) for $p \geq 5$ we have

$$(31) \quad H_{p-1}(2, 3) \equiv -2B_{p-5}, \quad H_{p-1}(4, 1) \equiv -B_{p-5} \pmod{p}.$$

Similarly, from (e) we obtain

$$H_{p-1}(2, 2, 1) \equiv \frac{3}{2}B_{p-5} \pmod{p}, \quad p > 5.$$

Finally, substituting the above congruences in (30), by Lemma 3, (g) and (24), we get the required congruence for all primes $p \geq 7$. The validity of Theorem 1 for $p = 5$ can be easily checked by straightforward verification. \square

Proof of Theorem 2.

From Theorem 1 and Lemma 1 we easily get the first congruence of Theorem 2. To prove the second one, which is dual to the first congruence, it is sufficient to apply [13, Th. 3.3] or simply note that for $k = 1, 2, \dots, p-1$ we have

$$\frac{p}{k \binom{2k}{k}} \equiv \frac{1}{2} \binom{2(p-k)}{p-k} \pmod{p}. \quad \square$$

Proof of Theorem 3.

For non-negative integers n, k define the pair of functions

$$F(n, k) = \frac{(-1)^k (n-k-1)! k!^2}{2(k+1)(n+k+1)!} H_k(2), \quad n \geq k+1,$$

$$G(n, k) = \frac{(-1)^k (n-k)! k!^2}{(n+k+1)!(n+1)^2} \left(H_k(2) - \frac{1}{(n+1)^2} \right), \quad n \geq k.$$

It is easy to check that the pair (F, G) is a WZ pair, i.e.,

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k)$$

for any $n, k \geq 0, n \geq k+1$. Applying formula (28) to (F, G) and replacing N by $N-1$ we get the identity

$$2 \sum_{k=1}^N \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^N \frac{(-1)^{k-1} H_{k-1}(2)}{k^3 \binom{2k}{k}} = \sum_{k=1}^N \frac{1}{k^5} + \frac{1}{2} \sum_{k=1}^N \frac{(-1)^k (N-k)! (k-1)!^2 H_{k-1}(2)}{k(N+k)!}.$$

Setting $N = p-1$ and employing (29) we get

$$2 \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1} H_{k-1}(2)}{k^3 \binom{2k}{k}} \equiv H_{p-1}(5) + \frac{1}{2} \sum_{k=1}^{p-1} \frac{H_{k-1}(2)}{k} \left(\frac{1}{pk} + \frac{1}{k^2} \right)$$

$$\equiv H_{p-1}(5) + \frac{1}{2p} H_{p-1}(2, 2) + \frac{1}{2} H_{p-1}(2, 3) \pmod{p}.$$

Now taking into account (24), (31) and the congruence [10, Cor. 5.1] $H_{p-1}(5) \equiv 0 \pmod{p}$ for $p > 5$, we get the required statement. The validity of Theorem 3 for $p = 5$ can be easily checked by hand. \square

Theorem 4 follows immediately from Theorem 3 and Lemma 2.

5. A p -ANALOGUE OF ZEILBERGER'S SERIES FOR $\zeta(2)$.

Let k be a non-negative integer. Define the sequence $\{b_{m,k}\}_{m \geq 0}$ by the power series expansion

$$\prod_{j=1}^k \left(1 + \frac{a}{j} \right)^{-2} = \sum_{m=0}^{\infty} b_{m,k} a^m \quad \text{if } k \geq 1, \quad |a| < 1,$$

and put $b_{0,0} = 1, b_{m,0} = 0, m \geq 1$.

Lemma 4. *For any non-negative integers m, k the sequence $\{b_{m,k}\}$ consists of rational numbers satisfying the following recurrence:*

$$(32) \quad b_{m,k} = \sum_{j=0}^m \frac{(-1)^j(j+1)}{k^j} b_{m-j,k-1}, \quad m \geq 0, \quad k \geq 1,$$

$$b_{0,k} = 1, \quad k \geq 0, \quad b_{m,0} = 0, \quad m \geq 1.$$

Moreover, for any prime $p > 3$ and a positive integer m we have

$$b_{m,p-1} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } m \text{ is odd;} \\ 0 \pmod{p}, & \text{if } m \text{ is even.} \end{cases}$$

The first few values of this sequence are as follows:

$$(33) \quad \begin{aligned} b_{1,k} &= -2H_k(1), & b_{2,k} &= 3H_k^2(1) - 2H_k(1,1), \\ b_{3,k} &= 6H_k(1)H_k(1,1) - 2H_k(1,1,1) - 4H_k^3(1), \\ b_{4,k} &= 5H_k^4(1) + 6H_k(1)H_k(1,1,1) + 3H_k^2(1,1) - 12H_k^2(1)H_k(1,1) - 2H_k(1,1,1,1). \end{aligned}$$

Proof. Observing that

$$\left(1 + \frac{a}{k}\right)^{-2} = \sum_{j=0}^{\infty} \frac{(-1)^j(j+1)}{k^j} a^j, \quad |a| < 1,$$

and multiplying the series

$$\left(1 + \frac{a}{k}\right)^{-2} \cdot \sum_{m=0}^{\infty} b_{m,k-1} a^m = \sum_{m=0}^{\infty} b_{m,k} a^m$$

we get the required recurrence (32). Expanding for $k \geq 1$,

$$\prod_{j=1}^k \left(1 + \frac{a}{j}\right)^{-1} = \left(\sum_{j=0}^k H_k(\{1\}^j) a^j\right)^{-1} = \sum_{m=0}^{\infty} c_{m,k} a^m, \quad |a| < 1,$$

and using the usual multiplication of the series we get

$$c_{0,k} = 1, \quad \sum_{j=0}^m H_k(\{1\}^j) c_{m-j,k} = 0, \quad m \geq 1,$$

which yields the recurrence formula for the coefficients $c_{m,k}$:

$$(34) \quad c_{0,k} = 1, \quad c_{m,k} = - \sum_{j=1}^m H_k(\{1\}^j) c_{m-j,k}, \quad m \geq 1.$$

From (34) it follows easily by induction on m that

$$(35) \quad c_{m,p-1} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } m \text{ is odd,} \\ 0 \pmod{p} & \text{if } m \text{ is even.} \end{cases}$$

Indeed, for $m = 1$ we have from (34) by Wolstenholme's theorem, $c_{1,p-1} = -H_{p-1}(1) \equiv 0 \pmod{p^2}$. If $m > 1$ is odd (even), then taking into account that the numbers j and $m - j$ have different (the same) parity and

$$H_{p-1}(\{1\}^j) \equiv \begin{cases} 0 \pmod{p^2} & \text{if } j \text{ is odd,} \\ 0 \pmod{p} & \text{if } j \text{ is even,} \end{cases}$$

by recurrence (34), we get the congruence (35). Since

$$(36) \quad b_{m,k} = \sum_{j=0}^m c_{j,k} c_{m-j,k},$$

by (35), we get the required congruences for $b_{m,p-1}$. Formulas (33) can be readily obtained from relations (34) and (36). \square

Proposition 1. *For any positive integers m, n we have*

$$\sum_{k=1}^n \frac{(3k-2)b_{m,k-1} + 2b_{m-1,k-1}}{k \binom{2k}{k}} = -\frac{b_{m,n}}{\binom{2n}{n}} + \sum_{k=1}^n \frac{3kb_{m-2,k} + 2b_{m-3,k}}{k^3 \binom{2k}{k}},$$

where $b_{-2,k} = b_{-1,k} := 0$, and

$$(37) \quad \sum_{k=1}^n \frac{3k-2}{k \binom{2k}{k}} = 1 - \frac{1}{\binom{2n}{n}}.$$

Proof. For a non-negative integer m consider the difference

$$(38) \quad \frac{b_{m,k}}{\binom{2k}{k}} - \frac{b_{m,k-1}}{\binom{2k}{k}} = \frac{b_{m,k}}{\binom{2k}{k}} - \frac{b_{m,k-1}}{\binom{2k-2}{k-1}} + \frac{(3k-2)b_{m,k-1}}{k \binom{2k}{k}}.$$

Summing (38) over $k = 1, 2, \dots, n$ we get

$$(39) \quad \sum_{k=1}^n \left(\frac{b_{m,k}}{\binom{2k}{k}} - \frac{b_{m,k-1}}{\binom{2k}{k}} \right) = \frac{b_{m,n}}{\binom{2n}{n}} - b_{m,0} + \sum_{k=1}^n \frac{(3k-2)b_{m,k-1}}{k \binom{2k}{k}}.$$

Putting $m = 0$ in (39) implies (37). If $m \geq 1$, then $b_{m,0} = 0$ and by the recurrence relation (32), we have

$$(40) \quad \frac{b_{m,n}}{\binom{2n}{n}} + \sum_{k=1}^n \frac{(3k-2)b_{m,k-1}}{k \binom{2k}{k}} = -\sum_{k=1}^n \frac{2b_{m-1,k-1}}{k \binom{2k}{k}} + \sum_{k=1}^n \sum_{j=2}^m \frac{(-1)^j (j+1)}{k^j \binom{2k}{k}} b_{m-j,k-1}.$$

If $m = 1$, then the double sum in (40) is empty and we get the desired identity. If $m \geq 2$, then we note that by (32), we have

$$\begin{aligned}
 (41) \quad 3kb_{m-2,k} + 2b_{m-3,k} &= 3k \sum_{j=0}^{m-2} \frac{(-1)^j(j+1)}{k^j} b_{m-2-j,k-1} + 2 \sum_{j=0}^{m-3} \frac{(-1)^j(j+1)}{k^j} b_{m-3-j,k-1} \\
 &= 3kb_{m-2,k-1} + \sum_{j=0}^{m-3} \frac{(-1)^j b_{m-3-j,k-1}}{k^j} (2j+2-3j-6) \\
 &= 3kb_{m-2,k-1} + \sum_{j=3}^m \frac{(-1)^j(j+1)b_{m-j,k-1}}{k^{j-3}} = k^3 \sum_{j=2}^m \frac{(-1)^j(j+1)b_{m-j,k-1}}{k^j}.
 \end{aligned}$$

Now from (40) and (41) we deduce the required statement. \square

Remark 5.1. Note that formula (37) also follows from [8, Th. 5.4].

Proposition 2. *Let $p > 5$ be a prime. Then the following congruences are true:*

$$\begin{aligned}
 p \sum_{k=1}^{p-1} \frac{3kb_{1,k} + 2b_{0,k}}{k^3 \binom{2k}{k}} &\equiv \frac{4H_{p-1}(1)}{p} \pmod{p^3}, \\
 p \sum_{k=1}^{p-1} \frac{3kb_{2,k} + 2b_{1,k}}{k^3 \binom{2k}{k}} &\equiv 0 \pmod{p^2}, \\
 p \sum_{k=1}^{p-1} \frac{3kb_{3,k} + 2b_{2,k}}{k^3 \binom{2k}{k}} &\equiv 0 \pmod{p}.
 \end{aligned}$$

Proof. For any non-negative integers n, k define the pair of functions

$$\begin{aligned}
 F(n, k) &= \frac{(-1)^{n+k} k!^2 n!^2 (1+a)_{n-k-1}}{(n+k+1)! (1+a)_n^2}, & n \geq k+1, \\
 G(n, k) &= \frac{(-1)^{n+k} k!^2 n!^2 (1+a)_{n-k} (2+2n+a)}{(n+k+1)! (1+a)_{n+1}^2}, & n \geq k,
 \end{aligned}$$

where $(a)_0 = 1$, $(a)_m = a(a+1) \cdots (a+m-1)$, $m \geq 1$, is the Pochhammer symbol. It is easy to check that the pair (F, G) satisfies (26) for any $n, k \geq 0$, $n \geq k+1$. Applying summation formula (28) and replacing N by $N-1$ we get the identity:

$$\begin{aligned}
 (42) \quad \sum_{k=1}^N \frac{(-1)^{k-1} \left(1 + \frac{k}{k+a}\right)}{k^2 \prod_{j=1}^k \left(1 + \frac{a}{j}\right)} &= \sum_{k=1}^N \frac{3k+2a}{k^3 \binom{2k}{k} \prod_{j=1}^k \left(1 + \frac{a}{j}\right)^2} \\
 &\quad + \prod_{j=1}^N \left(1 + \frac{a}{j}\right)^{-2} \cdot \sum_{k=1}^N \frac{(-1)^{N+k} (1+a)_{N-k} (k-1)!^2}{(N+k)!}.
 \end{aligned}$$

Now if we expand the summands in powers of a , it is easy to notice that

$$\sum_{k=1}^N \frac{3k+2a}{k^3 \binom{2k}{k} \prod_{j=1}^k (1 + \frac{a}{j})^2} = \sum_{m=0}^{\infty} \left(\sum_{k=1}^N \frac{3kb_{m,k} + 2b_{m-1,k}}{k^3 \binom{2k}{k}} \right) a^m.$$

Therefore, comparing coefficients of a^m on both sides of (42) leads to the following family of identities for all $m \geq 1$:

$$(43) \quad \sum_{k=1}^N \frac{3kb_{m,k} + 2b_{m-1,k}}{k^3 \binom{2k}{k}} = \sum_{k=1}^N \frac{(-1)^{k-1}}{k^3} \left(2k \sum_{j=0}^m b_{m-j,k} H_{k-1}(\{1\}^j) + \sum_{j=0}^{m-1} b_{m-1-j,k} H_{k-1}(\{1\}^j) \right) \\ - \sum_{k=1}^N \frac{(-1)^{N+k} (N-k)! (k-1)!^2}{(N+k)!} \sum_{j=0}^m H_{N-k}(\{1\}^j) b_{m-j,N},$$

where $H_n(\{1\}^0) := 1$. In particular, for $m = 1, 2$ we have

$$(44) \quad \sum_{k=1}^N \frac{3kb_{1,k} + 2}{k^3 \binom{2k}{k}} = \sum_{k=1}^N (-1)^k \left(\frac{2H_{k-1}(1)}{k^2} + \frac{3}{k^3} \right) \\ - \sum_{k=1}^N \frac{(-1)^{N+k} (N-k)! (k-1)!^2 (b_{1,N} + H_{N-k}(1))}{(N+k)!},$$

$$(45) \quad \sum_{k=1}^N \frac{3kb_{2,k} + 2b_{1,k}}{k^3 \binom{2k}{k}} = \sum_{k=1}^N \frac{(-1)^{k-1}}{k^2} \left(2H_k^2(1) - 2H_k(1, 1) + \frac{H_k(1)}{k} + \frac{1}{k^2} \right) \\ - \sum_{k=1}^N \frac{(-1)^{N+k} (N-k)! (k-1)!^2}{(N+k)!} (H_{N-k}(1, 1) + b_{1,N} H_{N-k}(1) + b_{2,N}).$$

Setting $N = p-1$ in (44) by (33) and (29), we obtain

$$p \sum_{k=1}^{p-1} \frac{3kb_{1,k} + 2b_{0,k}}{k^3 \binom{2k}{k}} \equiv 2pH_{p-1}(1, -2) + 3pH_{p-1}(-3) \\ - \sum_{k=1}^{p-1} \left(\frac{1}{k} + \frac{p}{k^2} + \frac{p^2 H_k(2)}{k} \right) (H_{p-1-k}(1) - 2H_{p-1}(1)) \pmod{p^3}.$$

Since

$$H_{p-1-k}(1) - H_{p-1}(1) = \frac{1}{1-p} + \cdots + \frac{1}{k-p} \equiv H_k(1) + pH_k(2) + p^2 H_k(3) \pmod{p^3},$$

by (a), (g), for $p > 5$ we have

$$p \sum_{k=1}^{p-1} \frac{3kb_{1,k} + 2b_{0,k}}{k^3 \binom{2k}{k}} \equiv \frac{3H_{p-1}(1)}{p} - \sum_{k=1}^{p-1} \frac{H_k(1)}{k} - p \sum_{k=1}^{p-1} \frac{H_k(2)}{k} - p \sum_{k=1}^{p-1} \frac{H_k(1)}{k^2} \\ - p^2 \sum_{k=1}^{p-1} \frac{H_k(3)}{k} - p^2 \sum_{k=1}^{p-1} \frac{H_k(2)}{k^2} - p^2 \sum_{k=1}^{p-1} \frac{H_k(2)H_k(1)}{k} \pmod{p^3}.$$

Applying (21) for evaluation of the last sum, by (a), we get

$$\begin{aligned} p \sum_{k=1}^{p-1} \frac{3kb_{1,k} + 2b_{0,k}}{k^3 \binom{2k}{k}} &\equiv \frac{3H_{p-1}(1)}{p} - H_{p-1}(1, 1) - H_{p-1}(2) - pH_{p-1}(2, 1) - pH_{p-1}(1, 2) \\ &\quad - 2p^2H_{p-1}(3, 1) - 2p^2H_{p-1}(2, 2) - p^2H_{p-1}(1, 3) \\ &\quad - p^2H_{p-1}(1, 2, 1) - p^2H_{p-1}(2, 1, 1) \pmod{p^3}. \end{aligned}$$

Now taking into account that

$$H_{p-1}(1, 1) = \frac{1}{2} \left(H_{p-1}^2(1) - H_{p-1}(2) \right) \equiv -\frac{H_{p-1}(2)}{2} \equiv \frac{H_{p-1}(1)}{p} \pmod{p^3}$$

by (a)–(f) and Lemma 3, we easily get the first congruence of the Proposition.

Similarly, setting $N = p - 1$ in (45) by (29), Lemma 4 and (a), we obtain

$$\begin{aligned} p \sum_{k=1}^{p-1} \frac{3kb_{2,k} + 2b_{1,k}}{k^3 \binom{2k}{k}} &\equiv p \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^2} \left(2H_k^2(1) - 2H_k(1, 1) + \frac{H_k(1)}{k} + \frac{1}{k^2} \right) \\ &\quad - \sum_{k=1}^{p-1} \left(\frac{1}{k} + \frac{p}{k^2} \right) \left(H_{p-1-k}(1, 1) - 2H_{p-1}(1, 1) \right) \pmod{p^2}. \end{aligned}$$

Applying (22) and the formula

$$H_k(1, 1) = H_{k-1}(1, 1) + \frac{H_{k-1}(1)}{k}$$

after simplifying we find that

$$\begin{aligned} p \sum_{k=1}^{p-1} \frac{3kb_{2,k} + 2b_{1,k}}{k^3 \binom{2k}{k}} &\equiv -2pH_{p-1}(1, 1, -2) - 3pH_{p-1}(1, -3) - 4pH_{p-1}(-4) \\ &\quad - 2pH_{p-1}(2, -2) - \sum_{k=1}^{p-1} \frac{H_{p-1-k}(1, 1)}{k} - p \sum_{k=1}^{p-1} \frac{H_{p-1-k}(1, 1)}{k^2} \pmod{p^2}. \end{aligned}$$

Since from (c), (f) it follows that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_{p-1-k}(1, 1)}{k} &= \sum_{k=1}^{p-1} \frac{H_{k-1}(1, 1)}{p-k} \equiv - \sum_{k=1}^{p-1} H_{k-1}(1, 1) \left(\frac{1}{k} + \frac{p}{k^2} \right) \\ &= -H_{p-1}(1, 1, 1) - pH_{p-1}(1, 1, 2) \equiv 0 \pmod{p^2} \end{aligned}$$

and

$$\sum_{k=1}^{p-1} \frac{H_{p-1-k}(1, 1)}{k^2} = \sum_{k=1}^{p-1} \frac{H_{k-1}(1, 1)}{(p-k)^2} \equiv \sum_{k=1}^{p-1} \frac{H_{k-1}(1, 1)}{k^2} = H_{p-1}(1, 1, 2) \equiv 0 \pmod{p},$$

we conclude, by (g), the second congruence of the Proposition.

Finally, from (43) with $m = 3$, $N = p - 1$, by (29), Lemma 4 and (a), (b), we have

$$p \sum_{k=1}^{p-1} \frac{3kb_{3,k} + 2b_{2,k}}{k^3 \binom{2k}{k}} \equiv - \sum_{k=1}^{p-1} \frac{H_{p-1-k}(\{1\}^3)}{k} \equiv \sum_{k=1}^{p-1} \frac{H_{k-1}(\{1\}^3)}{k} = H_{p-1}(\{1\}^4) \equiv 0 \pmod{p},$$

which completes the proof. \square

Proof of Theorem 5.

For any non-negative integers n, k define the pair of functions

$$F(n, k) = \frac{k!^4 n!^2 (2n + 3k + 3)}{2(2k + 1)!(k + n + 1)!^2}, \quad G(n, k) = \frac{k!^4 n!^2}{(2k)!(k + n + 1)!^2}.$$

It is straightforward to check that (F, G) is a WZ pair. Applying summation formula (28) and replacing N by $N - 1$ we get

$$\sum_{k=1}^N \frac{1}{k^2} = \sum_{k=1}^N \frac{21k - 8}{k^3 \binom{2k}{k}^3} - \sum_{k=1}^N \frac{k!(k-1)!^3 N!^2 (2N + 3k)}{(2k)!(k + N)!^2}.$$

Setting $N = p - 1$ and noting that for $1 \leq k \leq p - 1$,

$$\begin{aligned} \frac{k!(k-1)!^3 (p-1)!^2 (2p + 3k - 2)}{(2k)!(k + p - 1)!^2} &= \frac{2p + 3k - 2}{k \binom{2k}{k}} \frac{1}{p^2} \prod_{j=1}^{k-1} \left(1 + \frac{p}{j}\right)^{-2} \\ &\equiv \frac{2p + 3k - 2}{k \binom{2k}{k}} \sum_{m=0}^5 b_{m,k-1} p^{m-2} \pmod{p^3} \end{aligned}$$

we get

$$\begin{aligned} p^3 \sum_{k=1}^{p-1} \frac{21k - 8}{k^3 \binom{2k}{k}^3} &\equiv p^3 H_{p-1}(2) + p \sum_{k=1}^{p-1} \frac{3k - 2}{k \binom{2k}{k}} \\ &\quad + \sum_{m=1}^5 p^{m+1} \sum_{k=1}^{p-1} \frac{(3k - 2)b_{m,k-1} + 2b_{m-1,k-1}}{k \binom{2k}{k}} \pmod{p^6}. \end{aligned}$$

Now by Propositions 1, 2 and Lemma 4, for $p > 5$ we obtain

$$\begin{aligned} p^3 \sum_{k=1}^{p-1} \frac{21k - 8}{k^3 \binom{2k}{k}^3} &\equiv p^3 H_{p-1}(2) + p - \frac{p}{\binom{2p-2}{p-1}} - \sum_{m=1}^5 \left(\frac{b_{m,p-1}}{\binom{2p-2}{p-1}} - \sum_{k=1}^{p-1} \frac{3kb_{m-2,k} + 2b_{m-3,k}}{k^3 \binom{2k}{k}} \right) p^{m+1} \\ &\equiv p^3 H_{p-1}(2) + p - \frac{p}{\binom{2p-2}{p-1}} - \sum_{m=1}^5 \frac{b_{m,p-1} p^{m+1}}{\binom{2p-2}{p-1}} + 3 \sum_{k=1}^{p-1} \frac{p^3}{k^2 \binom{2k}{k}} + 4p^2 H_{p-1}(1) \pmod{p^6}. \end{aligned}$$

Employing (2), (b) and Lemma 4 we get

$$(46) \quad p^3 \sum_{k=1}^{p-1} \frac{21k - 8}{k^3 \binom{2k}{k}^3} \equiv p + 3p^2 H_{p-1}(1) - \frac{p}{\binom{2p-2}{p-1}} - \sum_{m=1}^4 \frac{b_{m,p-1} p^{m+1}}{\binom{2p-2}{p-1}} \pmod{p^6}.$$

Taking into account that

$$\begin{aligned} \frac{p}{\binom{2p-2}{p-1}} &= (2p - 1) \prod_{j=1}^{p-1} \left(1 + \frac{p}{j}\right)^{-1} \equiv 2p - 1 + pH_{p-1}(1) + p^2 H_{p-1}(1, 1) - 2p^2 H_{p-1}(1) \\ &\quad - 2p^3 H_{p-1}(1, 1) + p^3 H_{p-1}(1, 1, 1) + p^4 H_{p-1}(1, 1, 1, 1) \pmod{p^6} \end{aligned}$$

and

$$H_{p-1}(1, 1) = \frac{1}{2}(H_{p-1}^2(1) - H_{p-1}(2)) \equiv -\frac{1}{2}H_{p-1}(2) \equiv \frac{H_{p-1}(1)}{p} - \frac{1}{5}p^3 B_{p-5} \pmod{p^4},$$

by (33) and (c), we get

$$\begin{aligned} -\frac{p}{\binom{2p-2}{p-1}} &\equiv 1 - 2p - 2pH_{p-1}(1) + 4p^2H_{p-1}(1) + \frac{4}{5}p^5 B_{p-5} \pmod{p^6}, \\ -\frac{pb_{1,p-1}}{\binom{2p-2}{p-1}} &\equiv 4pH_{p-1}(1) - 2H_{p-1}(1) \pmod{p^5}, \\ -\frac{pb_{2,p-1}}{\binom{2p-2}{p-1}} &\equiv 4pH_{p-1}(1, 1) - 2H_{p-1}(1, 1) \equiv 4H_{p-1}(1) - \frac{2H_{p-1}(1)}{p} + \frac{2}{5}p^3 B_{p-5} \pmod{p^4}, \\ -\frac{pb_{3,p-1}}{\binom{2p-2}{p-1}} &\equiv -2H_{p-1}(1, 1, 1) \equiv \frac{4}{5}p^2 B_{p-5} \pmod{p^3}, \\ -\frac{p^2 b_{4,p-1}}{\binom{2p-2}{p-1}} &\equiv -2pH_{p-1}(1, 1, 1, 1) \equiv \frac{2}{5}p^2 B_{p-5} \pmod{p^3}. \end{aligned}$$

Now substituting the above congruences in (46) and simplifying we get the required statement. \square

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REFERENCES

- [1] J. M. Borwein, D. J. Broadhurst, J. Kamnitzer, Central binomial sums, multiple Clausen values, and zeta values, *Experiment. Math.* 10 (2001) 25–34.
- [2] D. H. Bailey, J. M. Borwein, D. M. Bradley, Experimental determination of Apéry-like identities for $\zeta(2n+2)$, *Experiment. Math.* 15 (2006) 281–289.
- [3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansion*, D. Reidel Publishing, Dordrecht, Holland, 1974.
- [4] Kh. Hessami Pilehrood, T. Hessami Pilehrood, Generating function identities for $\zeta(2n+2)$, $\zeta(2n+3)$ via the WZ method, *Electron. J. Combin.* 15 (2008) 9 pp.
- [5] M. E. Hoffman, Quasi-symmetric functions and \pmod{p} multiple harmonic sums, arXiv: math.NT/0401319 (2007).
- [6] M. Koecher, Letter (German), *Math. Intelligencer*, 2 (1979/80) 62–64.
- [7] D. Leshchiner, Some new identities for $\zeta(k)$, *J. Number Theory* 13 (1981) 355–362.
- [8] S. Mattarei, R. Tauraso, Congruences for central binomial sums and finite polylogarithms, arXiv:1012.1308v5 [math.NT]
- [9] A. van der Poorten, A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$. An informal report, *Math. Intelligencer* 1 (1978/79) 195–203.
- [10] Z. H. Sun, Congruences concerning Bernoulli numbers and Bernoulli polynomials, *Discrete Appl. Math.* 105 (2000) 193–223.
- [11] Z. W. Sun, A new series for π^3 and related congruences, arXiv:1009.5375v6 [math.NT]

- [12] Z. W. Sun, Super congruences and Euler numbers, *Sci. China Math.* 54 (2011) 2509–2535.
- [13] R. Tauraso, More congruences for central binomial coefficients, *J. Number Theory* 130 (2010) 2639–2649.
- [14] R. Tauraso, J. Zhao, Congruences of alternating multiple harmonic sums, *J. Comb. Number Theory* 2 (2010) 129–159; arXiv:0909.0670 [math.NT]
- [15] J. Zhao, Wolstenholme type theorem for multiple harmonic sums, *Int. J. Number Theory* 4 (2008) 73–106.
- [16] X. Zhou, T. Cai, A generalization of a curious congruence on harmonic sums, *Proc. Amer. Math. Soc.* 135 (2007) 1329–1333.

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